Corrigé - Examen de fin d'études secondaires techniques - Mathématiques II

QUESTION 1

a) $u_1 = \dfrac{2u_0+3}{u_0+4} = \dfrac{3}{4}$

Si $u_n > 0$, alors $2u_n + 3 > 0$ et $u_n + 4 > 0$, donc $u_{n+1} = \dfrac{2u_n+3}{u_n+4} > 0$.

Par récurrence, $u_n > 0$, $\forall n \geq 1$.

b) $v_{n+1} = \dfrac{u_{n+1}-1}{u_{n+1}+3} = \dfrac{\dfrac{2u_n+3}{u_n+4}-1}{\dfrac{2u_n+3}{u_n+4}+3} = \dfrac{2u_n+3-u_n-4}{2u_n+3+3u_n+12} = \dfrac{u_n-1}{5(u_n+3)} = \dfrac{1}{5}v_n$

(v_n) est donc une suite géométrique de premier terme $v_0 = \dfrac{u_0-1}{u_0+3} = \dfrac{-1}{3}$ et de raison

$q = \dfrac{1}{5}$.

c) $v_n = -\dfrac{1}{3}\cdot\left(\dfrac{1}{5}\right)^n$ et $\lim\limits_{n\to\infty} v_n = 0$.

$v_n = \dfrac{u_n-1}{u_n+3} \Rightarrow u_n v_n + 3v_n = u_n - 1 \Rightarrow u_n(v_n-1) = -1-3v_n \Rightarrow u_n = \dfrac{1+3v_n}{1-v_n}$

$\lim\limits_{n\to\infty} u_n = 1$

QUESTION 2

a) $M(z) \in E$

$\Leftrightarrow (\bar{z}-3)\cdot(iz+2) \in \mathbb{R}$

$\Leftrightarrow (x-yi-3)\cdot(ix-y+2) \in \mathbb{R}$

$\Leftrightarrow ix^2 - xy + 2x + xy + iy^2 - 2yi - 3xi + 3y - 6 \in \mathbb{R}$

$\Leftrightarrow x^2 + y^2 - 3x - 2y = 0$

$\Leftrightarrow x^2 - 3x + \dfrac{9}{4} + y^2 - 2y + 1 = \dfrac{13}{4}$

$\Leftrightarrow \left(x-\dfrac{3}{2}\right)^2 + (y-1)^2 = \left(\dfrac{\sqrt{13}}{2}\right)^2$

E est un cercle de centre $\Omega\left(\dfrac{3}{2};1\right)$ et de rayon $r = \dfrac{\sqrt{13}}{2}$.

b) $$1+e^{i\frac{2\pi}{3}}=1+\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)$$
$$=2\cos^2\left(\frac{\pi}{3}\right)+i\cdot 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$$
$$=2\cos\left(\frac{\pi}{3}\right)\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$$
$$=e^{i\frac{\pi}{3}}$$

c) $$\cos^3 x\sin x=\left(\frac{e^{ix}+e^{-ix}}{2}\right)^3\cdot\frac{e^{ix}-e^{-ix}}{2i}$$
$$=\frac{1}{16i}\cdot\left(e^{i3x}+3e^{ix}+3e^{-ix}+e^{-i3x}\right)\left(e^{ix}-e^{-ix}\right)$$
$$=\frac{1}{16i}\cdot\left(e^{i4x}+3e^{i2x}+3+e^{-i2x}-e^{i2x}-3-3e^{-i2x}-e^{-i4x}\right)$$
$$=\frac{1}{16i}\cdot\left(e^{i4x}-e^{-i4x}+2\left(e^{i2x}-e^{-i2x}\right)\right)$$
$$=\frac{1}{8}\cdot\frac{e^{i4x}-e^{-i4x}}{2i}+\frac{1}{4}\cdot\frac{e^{i2x}-e^{-i2x}}{2i}$$
$$=\frac{1}{8}\sin(4x)+\frac{1}{4}\sin(2x)$$

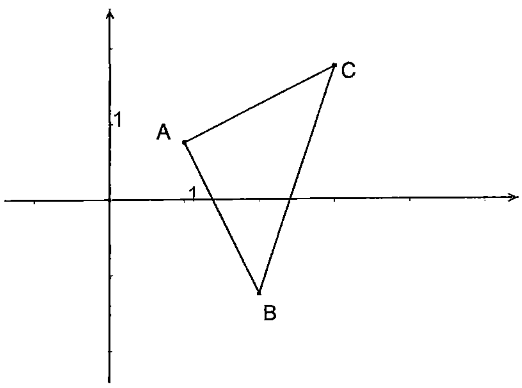
Sans les formules d'Euler :

$$\cos^3 x\cdot\sin x=\cos^2 x\cdot\cos x\cdot\sin x$$
$$=\frac{1+\cos(2x)}{2}\cdot\frac{1}{2}\cdot\sin(2x)$$
$$=\frac{1}{4}\left(\sin(2x)+\sin(2x)\cdot\cos(2x)\right)$$
$$=\frac{1}{4}\left(\sin(2x)+\frac{1}{2}\cdot\sin(4x)\right)$$
$$=\frac{1}{4}\sin(2x)+\frac{1}{8}\sin(4x)$$

QUESTION 3

$z_A=1+\frac{3}{4}i$; $z_B=2-\frac{5}{4}i$; $z_C=3+\frac{7}{4}i$

a)



$$z_{\overline{AB}} = z_B - z_A = 2 - \frac{5}{4}i - 1 - \frac{3}{4}i = 1 - 2i$$

$$z_{\overline{AC}} = z_C - z_A = 3 + \frac{7}{4}i - 1 - \frac{3}{4}i = 2 + i$$

$$z_{\overline{BC}} = z_C - z_B = 3 + \frac{7}{4}i - 2 + \frac{5}{4}i = 1 + 3i$$

b) $AB = \sqrt{1+4} = \sqrt{5}$

$AC = \sqrt{4+1} = \sqrt{5}$

$BC = \sqrt{1+9} = \sqrt{10}$

c) Comme $BC^2 = AB^2 + AC^2$, le triangle ABC est rectangle en A. Comme $AB = AC$, il est de plus isocèle en A.

d) $z_G = \frac{1}{3}(z_A + z_B + z_C) = \frac{1}{3}\left(6 + \frac{5}{4}i\right) = 2 + \frac{5}{12}i$

QUESTION 4

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QUESTION 5

a) $\frac{a}{x} + \frac{b}{1+x} + \frac{c}{(1+x)^2} = \frac{a(1+x)^2 + bx(1+x) + cx}{x(1+x)^2} = \frac{(a+b)x^2 + (2a+b+c)x + a}{x(1+x)^2}$

$$f(x) = \frac{(a+b)x^2 + (2a+b+c)x + a}{x(1+x)^2}, \forall x \in \mathbb{R} \setminus \{-1;0\}$$

$$\Leftrightarrow \begin{cases} a+b=0 \\ 2a+b+c=0 \\ a=1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=1 \\ b=-1 \\ c=-1 \end{cases}$$

D'où : $f(x) = \frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2}$.

$$\int_1^4 f(x)\,dx = \int_1^4 \left(\frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} \right) dx$$

$$= \left[\ln|x| - \ln|1+x| + \frac{1}{1+x} \right]_1^4$$

$$= \ln 4 - \ln 5 + \frac{1}{5} + \ln 2 - \frac{1}{2}$$

$$= \ln\frac{8}{5} - \frac{3}{10}$$

b)

$$\int_0^{\frac{\pi}{2}} x^2 \cos(2x)\,dx \qquad \left\| \begin{array}{ll} u = x^2 & v' = \cos(2x) \\ u' = 2x & v = \dfrac{1}{2}\sin(2x) \end{array} \right.$$

$$= \left[\frac{1}{2}x^2 \sin(2x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x \sin(2x)\,dx$$

$$= 0 + \int_{\frac{\pi}{2}}^{0} x \sin(2x)\,dx \qquad \left\| \begin{array}{ll} u = x & v' = \sin(2x) \\ u' = 1 & v = -\dfrac{1}{2}\cos(2x) \end{array} \right.$$

$$= \left[-\frac{1}{2}x \cos(2x)\right]_{\frac{\pi}{2}}^{0} + \frac{1}{2}\int_{\frac{\pi}{2}}^{0} \cos(2x)\,dx$$

$$= -\frac{\pi}{4} + \frac{1}{2}\left[\frac{1}{2}\sin(2x)\right]_{\frac{\pi}{2}}^{0}$$

$$= -\frac{\pi}{4} + \frac{1}{2}(0-0)$$

$$= -\frac{\pi}{4}$$

QUESTION 6

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QUESTION 7

$$M(x;y) \in Q$$

$$\Leftrightarrow \overrightarrow{BM}\begin{pmatrix} x-3 \\ y \\ z+2 \end{pmatrix} \perp \vec{n}\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

$$\Leftrightarrow 2x-6-y+5z+10=0$$

$$\Leftrightarrow 2x-y+5z+4=0$$